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Solutions of Gravitational Field Equations

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In this paper we use iterative methods to generate series solutions of the gravitational field equations in a cosmological model with heat flow.

The Einstein field equations, which relate matter to geometry (Misner *et al.*, 1973; Stephani, 1982), are expressed in terms of the space-time metric $g_{ab}(x^c)$, its partial derivatives up to second order, and the product of these derivatives. This makes this coupled system of partial differential equations (pde) highly nonlinear and it becomes a difficult problem to seek its exact solutions. However, one can deal with this problem by requiring that the space-time metric possesses some isometries (the isometries are representative of Killing vectors and k is called a Killing vector if the Lie derivative of the metric tensor remains zero along that k). The number of Killing vectors possessed by different space-times corresponds to physical situations representing axial, spherical, and Minkowski symmetries, etc. (Bokhari, 1992a,b; Bergman, 1975, 1981; Glass, 1979). The greater is the number of Killing vectors, the greater is the symmetry and much easier is the task to obtain solutions of the gravitational field equations. A complete classification of these solutions according to their isometries is available (Petrov, 1969; Kramer *et al.*, 1980; Bokhari and Qadir, 1987, 1988, 1990; Bokhari *et al.*, 1993).

While isometries give rise to solutions to the gravitational field equations, they impose certain important physical restrictions on the stress-energy tensor, e.g., if one chooses the Einstein field equations to define the energy-momentum tensor, then all metrics, no matter how arbitrary they are, would become solutions of these equations yielding arbitrary energy-momentum tensor

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fields, which may correspond to physically uninteresting models. Since not all models can be allowed, there arises a need to constrain these solutions by imposing constraints on the energy-momentum tensor fields, as a consequence of which one requires that the space-times possess isometries.

In this paper we address the problem of seeking the solutions of a general Robertson–Walker cosmological model presented earlier (Bokhari, 1992a,b). The metric of this model does not possess a time-translational isometry, but admits of spatial Killing vectors only. Apart from deriving a pressure isotropy condition and an expression for heat flow for completeness, we present a derivation of series solutions of the gravitational field equations in this model. The space-time metric of the cosmological model considered is

$$ds^2 = A^2(t, r)\, dt^2 - B^2(t, r)[(1 - k^2/R^2)^{-1}dr^2 + r^2\, d\Omega^2] \tag{1}$$

where k corresponds to the three Friedmann cosmological models for $A = 1$ in the above equation and

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\phi^2$$

The isotropic energy-momentum tensor field associated with the above metric is

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab} + 2q^{(a}u^{b)} \tag{2}$$

where ρ, p, and q^a, respectively, represent mass-energy density, isotropic pressure, and the radial component of the heat flow vector. The u^a represents the unit 4-velocity vector of the fluid defined by

$$u^a = A^{-1}\delta_0^a \tag{3}$$

Writing the Einstein field equations (in the units in which $c = 1 = \hbar = G$) in the form in which R_{ab} is used to derive the matter content of the given space-time,

$$R_{ab} = \kappa(T_{ab} - \tfrac{1}{2}g_{ab}T) \tag{4}$$

and solving for $a = 0$ and $b = 1$ yields the radial component of heat flow vector,

$$q^a = \frac{2(1 - k^2/R^2)}{\kappa}\,\overline{B}^2\left[\frac{\dot{B}}{AB}\right]'\delta_1^a \tag{5}$$

where $\kappa = 8\pi$, T is the trace of the energy momentum field, and the dot and prime, respectively, represent differentiation with respect to temporal and radial coordinates. Notice that the Einstein field equations [see equation (4)] may give rise to any arbitrary energy-momentum tensor, which may not correspond to a physically plausible solution. Thus we require that the energy-momentum tensor be isotropic, i.e.,

$$R_1^1 = R_2^2 = R_3^3 \tag{6}$$

Using the transformation $x = r^2$, we find that the isotropy condition gives rise to an interesting second-order pde in derivatives of x only,

$$(FA'' + 2F'A' - AF'')(1 - kx) - \frac{k}{2}(A'F - F'A) = 0 \tag{7}$$

where we have used $B = (1/F)$ in equation (7) and the prime now represents derivative with respect to the x coordinate.

To be able to generate series solutions of equation (7), we initially suppose that $A = A_1 = 1$, and $F = F_1$ are solutions of that equation and that F is an arbitrary function of time and radial coordinate. Under these assumptions equation (7) yields a second-order pde in F only given by

$$F_1''(1 - kx) + k/2\, F_1' = 0 \tag{8}$$

The above equation can be easily solved to yield

$$F_1 = a(t)(1 - kx)^{1/2} + b(t) \tag{9}$$

where $b(t)$ is an integration function and $a(t)$, which is given by $-2c(t)/k$, equals a function of integration, $c(t)$, times $-2/k$ with k being nonzero.

Now rewriting equation (9) in the form $F_1 = a\alpha + b$, where $\alpha = (1 - kx)^{1/2}$, and using it in equation (7) yields

$$\frac{A_2''}{A_2} + \frac{2a\alpha'}{a\alpha + b} - \frac{k}{2(1 - kx)} = 0 \tag{10}$$

The above equation is a second-order pde in A_2 and its derivatives, which can also be easily solved to obtain

$$A_2 = \frac{d\alpha + e}{a\alpha + b} \tag{11}$$

where $d = d(t)$ and $e = e(t)$ are again functions of integration depending only on the time coordinate.

Up to now we have found, using iterative methods, assuming $A = 1$ initially, F_1 and A_2. The next step is to find F_2 given A_2. For this the isotropy condition gives rise to a pde

$$V'' + \frac{2a\alpha' V'}{a\alpha + b} - \frac{2(bd - ae)}{(a\alpha + b)(d\alpha + e)}\alpha' V' - \frac{kV'}{2\alpha^2} = 0 \qquad (12)$$

where we have substituted $V = F_2/(a\alpha + b)$. Setting $V' = V_1$ in equation (12), we can write the same equation in the convenient form

$$\frac{V_1'}{V_1} = \frac{-2a\alpha'}{a\alpha + b} + \frac{2(bd - ae)}{(a\alpha + b)(d\alpha + e)} + \frac{k}{2\alpha^2} \qquad (13)$$

After some tedious calculations the above equation can be easily solved for V_1. Rearranging terms and transforming back to F_2, for the solution of equation (13), we obtain

$$F_2 = \frac{2}{3ak}\left[\left(\frac{d\alpha + e}{a\alpha + b}\right)^2 + \frac{d}{a}\left(\frac{d\alpha + e}{a\alpha + b}\right) + \frac{d^2}{a^2}\right] \qquad (14)$$

Having obtained solutions in terms of A_1, A_2, F_1, and F_2, we can finally write the general solution of equation (7) as

$$F(x, t) = g(t)(a\alpha + b) + \frac{2h(t)}{3ak}\left[\left(\frac{d\alpha + e}{a\alpha + b}\right)^2 + \frac{d}{a}\left(\frac{d\alpha + e}{a\alpha + b}\right) + \frac{d^2}{a^2}\right]$$

This procedure can be continued and the next solutions can be easily generated. In the special case in which $k = 0$, these solutions reduce to those of Deng (1989).

Note that the model considered here differs from the one considered by Deng (1989) by the term $(1 - kr^2/R^2)^{-1}$ in the one–one component of the metric. This term, however, matters only when k is considered in the context of a closed Friedmann model, where the radial component of heat flow can be considered meaningful at least at the very early stages of the closed Friedmann era or it could be of interest when the closed Friedmann model converges to the big crunch singularity (at least at the classical level). In the open case the radial component of heat flow is of no interest whatsoever, hence the above analysis is restricted only to the closed Robertson-type cosmologies.

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